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H. SOLOMON and C. SUTTON

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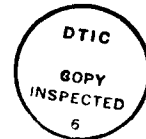
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# A SIMULATION STUDY OF RANDOM CAPS ON A SPHERE

H. Solomon and C. Sutton

**Abstract:** This paper describes the computer simulation of a coverage problem in geometric probability, that of placing random caps on the surface of a sphere. The simulation results were compared with exact values where known and the differences were negligible. This suggested the use of simulation results to assess several approximation formulas in the literature.

## 1. Introduction

Consider a sphere of unit radius on which are placed  $N$  spherical caps subtending a half angle  $\alpha$  at the center of the sphere ( $0 < \alpha \leq \pi/2$ ). Assume that the centers of these caps are independently and uniformly distributed over the surface of the sphere. We seek to approximate the probability  $P(N)$  that the sphere is completely covered. An exact expression for  $P(N)$  is presently not known except for several special cases, see Solomon (1978).

An approximation of  $P(N)$  can be obtained via computer simulation as follows. For each of  $n$  trials we generate  $N$  random caps on the surface of a unit sphere. We let  $m_{N,n}$  denote the number of trials in which the caps completely cover the sphere. Then  $p_{N,n} = m_{N,n}/n$  is an approximate value of  $P(N)$ . As  $n$  increases, the approximation should approach the exact value.

In order to determine whether or not the caps of a given trial completely cover a sphere, let us define a crossing as being a point of intersection of the circular boundaries of two overlapping caps. Then if there is an uncovered crossing, some area of the sphere outside the two overlapping caps must also be uncovered. Conversely, if at least two caps overlap, then the boundary of any uncovered area on the sphere must contain an uncovered crossing.

Thus we find that the sphere is completely covered if and only if there are at least two caps which overlap and every crossing is covered. This is the essential fact on which the simulation program is based.

Gilbert (1965) used the preceding idea to perform a similar simulation; however, his study was rather small and the results do not provide enough data with which to evaluate the accuracy of the various approximation formulas which have been suggested for  $P(N)$ . Prior results were also achieved by Moran and Fazekas de St. Groth (1962) by employing a small physical simulation. We desire to assess the accuracy of the suggested approximation formulas and have therefore performed a large computer simulation. Some of the main steps incorporated in our simulation program are described below.

## 2. Simulation

Random points on the surface of the sphere, which provide centers for the caps, are obtained as follows. For each point  $p_i$  we first generate  $q_{i,1}$ ,  $q_{i,2}$ , and  $q_{i,3}$  randomly, each according to a normal distribution with mean 0 and variance 1. We then perform a projection by letting

$$\begin{aligned} p_{i,1} &= \frac{q_{i,1}}{\|q\|} \\ p_{i,2} &= \frac{q_{i,2}}{\|q\|} \\ p_{i,3} &= \frac{q_{i,3}}{\|q\|} \end{aligned}$$

where

$$\|q\| = (q_{i,1}^2 + q_{i,2}^2 + q_{i,3}^2)^{\frac{1}{2}}.$$

Due to the symmetry of the spherical normal distribution, the point  $p_i = (p_{i,1}, p_{i,2}, p_{i,3})$  so obtained will be a random observation from a distribution which is uniformly distributed over the surface of the sphere.

A cap intersects another cap if its center falls within a great circle distance  $2\alpha$  from the center of the other cap. Letting  $d(a, b)$  denote the straight line distance between two points  $a$  and  $b$ , we then have that the cap with center  $a$  intersects the cap with center  $b$  if and only if

$$\arcsin(d(a, b)/2) < \alpha.$$

Suppose it occurs that the cap with center  $a = (a_1, a_2, a_3)$  intersects the cap with center  $b = (b_1, b_2, b_3)$ . Then two crossings are formed, one on each side of the great circle arc connecting points  $a$  and  $b$ . Denoting the midpoint of this arc by  $m$ , the crossings will lie on the great circle which forms a perpendicular intersection at the point  $m$  with the great circle passing through  $a$  and  $b$ . Using spherical trigonometry we obtain that the great circle distance between each crossing and the point  $m$  is given by

$$\arccos \left\{ \frac{\cos \alpha}{\cos \left[ \arcsin \frac{d(a,b)}{2} \right]} \right\} = \arccos \left\{ \frac{\cos \alpha}{\left[ 1 - \frac{d^2(a,b)}{4} \right]^{\frac{1}{2}}} \right\}.$$

Denote this distance by  $\gamma$ , and call the two crossings formed  $x$  and  $y$ .

We can determine the Cartesian coordinates  $(x_1, x_2, x_3)$  of  $x$  and  $(y_1, y_2, y_3)$  of  $y$  in the following manner. Set  $v = (v_1, v_2, v_3)$  equal to  $(b_1 - a_1, b_2 - a_2, b_3 - a_3)$ . Let  $\|v\|$  denote the length of the vector  $v$ . Let  $\theta$  be the angle between the positive x-axis and the vector  $(v_1, v_2, 0)$ , measured in the direction from the positive x-axis towards the positive y-axis. Let  $\phi = \arcsin(v_3/\|v\|)$ . Consider the following series of rotation of axes. First rotate about the z-axis an angle  $\theta$  in the direction of  $(v_1, v_2, 0)$ . Then rotate about the new y-axis an angle  $\phi$  towards the positive z-axis. Next we rotate about the new x-axis an angle  $\gamma$  towards the positive y-axis. Now if we reverse the direction of the first two rotations, and rotate about the y-axis by  $\phi$  and then about the z-axis by  $\theta$ , the new coordinates of  $m$  will be the coordinates of the crossing  $x$  in the original system. This sequence of rotations of axes is equivalent to rotating the sphere through an angle  $\gamma$  about an axis that passes through the center of the sphere and which is parallel to the vector  $v$ . The coordinates of  $y$  are found in the same way, except that we rotate about the x-axis by an angle  $-\gamma$ .

A crossing formed by two caps will be covered if the great circle distance between the crossing and the center of any of the  $N-2$  other caps is less than  $\alpha$ . Equivalently, we can check whether or not the straight line distance is less than  $2 \sin(\alpha/2)$ .

Simulations were done using  $\alpha = \pi/2$  and  $N = 4, 5, \dots, 13$ . This value of  $\alpha$  was selected since exact values for  $P(N)$  have been determined in this case and hence can serve as anchors in our simulation study. J. G. Wendel (1962) showed that if  $N$  points are scattered at random on the surface of the unit sphere in  $n$ -space, the probability that all the points lie on some

hemisphere is given by

$$2^{-N+1} \sum_{k=0}^{n-1} \binom{N-1}{k}.$$

This expression yields, as a special case, the result that for spherical caps of angular radius  $\alpha = \pi/2$

$$(1) \quad P(N) = 1 - 2^{-N}(N^2 - N + 2).$$

For each value of  $N$  20,000 trials were accomplished. The number of covered spheres out of 20,000 trials,  $m_{N,20000}$ , is binomially distributed with parameters  $n = 20,000$  and  $p = P(N)$ . Hence the standard deviation of  $p_{N,20000}$  ( $= m_{N,20000}/20000$ ) is given by

$$SD_{N,20000} = \sqrt{\frac{P(N)(1 - P(N))}{20000}}.$$

The following results give, for each value of  $N$ , the exact value of  $P(N)$  for  $\alpha = \pi/2$ , our approximation  $p_{N,20000}$ , and the value of  $r_{N,20000} = (p_{N,20000} - P(N))/SD_{N,20000}$ .

| $N$ | $P(N)$  | $p_{N,20000}$ | $r_{N,20000}$ |
|-----|---------|---------------|---------------|
| 4   | 0.12500 | 0.12550       | 0.21          |
| 5   | 0.31250 | 0.31555       | 0.93          |
| 6   | 0.50000 | 0.50030       | 0.08          |
| 7   | 0.65625 | 0.66055       | 1.28          |
| 8   | 0.77344 | 0.77145       | -0.67         |
| 9   | 0.85547 | 0.85720       | 0.70          |
| 10  | 0.91016 | 0.91100       | 0.42          |
| 11  | 0.94531 | 0.94430       | -0.63         |
| 12  | 0.96729 | 0.96825       | 0.76          |
| 13  | 0.98071 | 0.98090       | 0.20          |

These results serve to validate our simulation program.

### 3. Moran and Fazekas de St. Groth results

We desire to investigate the various formulas that have been developed to approximate  $P(N)$ . First the formulas will be compared for the case of  $\alpha = 53.43^\circ$ . This value of  $\alpha$  is chosen since it has been considered by previous investigators (because of a biological application). To serve as a basis for the comparisons we estimated  $P(N)$  using our simulation program for



$N = 10, 15, 20, \dots, 60$  ( and  $\alpha = 53.43^\circ$ ). Each estimate is based on 20,000 trials. We note that even for 20,000 trials the standard deviation of the estimator  $p_{N,20000}$  can be as large as 0.0035 for some values of  $N$ . Hence the displayed simulation results in this paper may indicate more accuracy than is actually present.

Moran and Fazekas de St. Groth (1962) used a method of moments approach to derive two approximation formulas for  $P(N)$ . Under the assumption that the uncovered region of the sphere consists of a single region (which will always be the case when  $\alpha = \pi/2$  and should be nearly true for  $\alpha$  close to  $\pi/2$ ) they developed

$$(2) \quad P(N) = 1 - \frac{1}{2} \pi^2 [E(Y)]^2 / E(Y^2),$$

where  $Y$  denotes the proportion of the surface not covered. It is found that

$$E(Y) = \cos^{2N}(\alpha/2)$$

and the calculation of  $E(Y^2)$  may be accomplished via numerical integration using

$$(3) \quad E(Y^2) = \frac{1}{2} \int_0^\pi \left[ 1 - \frac{1}{4\pi} f(\phi) \right]^N \sin \phi \, d\phi$$

and

$$(4) \quad 1 - \frac{f(\phi)}{4\pi} = \begin{cases} \left[ 1 - \frac{1}{\pi} \arccos\left(\frac{\tan(\phi/2)}{\tan \alpha}\right) \right] \cos \alpha + \frac{1}{\pi} \arccos\left(\frac{\sin(\phi/2)}{\sin \alpha}\right) & (0 \leq \phi \leq 2\alpha) \\ \cos \alpha & (2\alpha \leq \phi \leq \pi). \end{cases}$$

Under the hypothesis that the number of disjoint uncovered regions has a Poisson distribution and that the areas of the individual uncovered regions are distributed independently, Moran and Fazekas de St. Groth derived

$$(5) \quad P(N) = \exp \left( -\frac{1}{2} \pi^2 \{ E(Y^2) [E(Y)]^{-2} - 1 \}^{-1} \right).$$

They expected that the actual value of  $P(N)$  should be between the values given by (2) and (5).

In the table below we give our simulation results for  $P(N)$  for  $\alpha = 53.43^\circ$  based on 20,000 trials for each value of  $N$ , as well as the values obtained from both of the formulas of Moran and Fazekas de St. Groth.

| $N$ | Simulation | (2),(3)&(4) | (5),(3)&(4) |
|-----|------------|-------------|-------------|
| 10  | 0.0003     | -2.6119     | 0.0000      |
| 15  | 0.0465     | -1.3333     | 0.0120      |
| 20  | 0.2752     | -0.2882     | 0.1750      |
| 25  | 0.5763     | 0.3643      | 0.4821      |
| 30  | 0.7846     | 0.7085      | 0.7336      |
| 35  | 0.8993     | 0.8738      | 0.8785      |
| 40  | 0.9556     | 0.9474      | 0.9482      |
| 45  | 0.9837     | 0.9787      | 0.9788      |
| 50  | 0.9929     | 0.9915      | 0.9916      |
| 55  | 0.9974     | 0.9967      | 0.9967      |
| 60  | 0.9988     | 0.9987      | 0.9987      |

It appears that both approximation formulas converge to a common value as  $N$  increases; however, they do not sandwich the simulation value as was anticipated by Moran and Fazekas de St. Groth. Rather, both formulas tend to underestimate  $P(N)$ .

In their paper, Moran and Fazekas de St. Groth did not use (3) and (4) in their calculations of  $P(N)$ . Instead they use a saddlepoint method to arrive at

$$(6) \quad \hat{m}_2 = \frac{1}{2} E(Y) \left[ 1 + \frac{N^2 \tan^2(\alpha/2)}{\pi^2} \right]^{-1}$$

as an approximation of  $E(Y^2)$ . It is interesting to note that making use of this substitution leads to values of  $P(N)$  which better approximate these simulation values. A comparison is presented in the table below, where for each case it can be seen that the approximation (6) yields a better estimate.

| $N$ | Simulation value | Value from (2),(3)&(4) | Value from (2) and (6) | Value from (5),(3)&(4) | Value from (5) and (6) |
|-----|------------------|------------------------|------------------------|------------------------|------------------------|
| 10  | 0.0003           | -2.6119                | -2.6814                | 0.0000                 | 0.0000                 |
| 15  | 0.0465           | -1.3333                | -1.2616                | 0.0120                 | 0.0154                 |
| 20  | 0.2752           | -0.2882                | -0.2163                | 0.1750                 | 0.1991                 |
| 25  | 0.5763           | 0.3643                 | 0.4050                 | 0.4821                 | 0.5084                 |
| 30  | 0.7846           | 0.7085                 | 0.7279                 | 0.7336                 | 0.7498                 |
| 35  | 0.8993           | 0.8787                 | 0.8815                 | 0.8785                 | 0.8857                 |
| 40  | 0.9556           | 0.9474                 | 0.9503                 | 0.9482                 | 0.9511                 |
| 45  | 0.9837           | 0.9787                 | 0.9798                 | 0.9788                 | 0.9799                 |
| 50  | 0.9929           | 0.9915                 | 0.9920                 | 0.9916                 | 0.9920                 |
| 55  | 0.9974           | 0.9967                 | 0.9969                 | 0.9967                 | 0.9969                 |
| 60  | 0.9988           | 0.9987                 | 0.9988                 | 0.9987                 | 0.9988                 |

For  $\alpha = 53.43^\circ$  and  $N = 10, 15, \dots, 60$  we found that the combination of (5) and (6) constituted the best approximation for  $P(N)$  produced by Moran and Fazekas de St. Groth.

#### 4. Gilbert developement

E. N. Gilbert (1965) developed upper and lower bounds for  $P(N)$ . Let  $\lambda$  denote the probability that any specified point on the sphere will be covered by a randomly placed cap, so that

$$\lambda = \frac{1}{2}(1 - \cos \alpha) = \sin^2(\alpha/2).$$

The lower bound for  $P(N)$  is given by

$$(7) \quad 1 - \frac{4}{3}N(N-1)\lambda(1-\lambda)^{N-1}.$$

Gilbert also gives

$$(8) \quad 1 - (1 - \lambda)^N$$

as a general upper bound for  $P(N)$ . This is a weak upper bound. For the specific case of  $\alpha = 53.43^\circ$  a better upper bound can be found. The dominant terms of this closer upper bound are

$$(9) \quad 1 - 6(1 - \lambda)^N.$$

Note that (8) is just the probability that some fixed point  $V$  on the sphere is covered, and notice that the probability that  $V$  is covered exceeds the probability that the entire sphere is covered. To arrive at (9), consider points  $V_1, V_2, \dots, V_6$  which are points of intersection of the sphere and of a regular octahedron inscribed within the sphere. The method of inclusion and exclusion can be used to find an exact expression for the probability that the set of six vertices is covered by  $N$ . This probability will be an upper bound for  $P(N)$ . For large  $N$ , (9) gives the dominant terms of this expression. It can be seen from the table below that while the upper and lower bounds do indeed bracket the simulation values, neither provides a very good estimate of  $P(N)$ .

| $N$ | lower bound (7) | simulation | upper bound (9) |
|-----|-----------------|------------|-----------------|
| 10  | -2.1790         | 0.0003     | 0.3725          |
| 15  | -1.3989         | 0.0465     | 0.7970          |
| 20  | -0.4039         | 0.2752     | 0.9344          |
| 25  | 0.2831          | 0.5763     | 0.9788          |
| 30  | 0.6638          | 0.7846     | 0.9931          |
| 35  | 0.8513          | 0.8993     | 0.9978          |
| 40  | 0.9370          | 0.9556     | 0.9993          |
| 45  | 0.9741          | 0.9837     | 0.9998          |
| 50  | 0.9896          | 0.9929     | 0.9999          |
| 55  | 0.9959          | 0.9974     | 1.0000          |
| 60  | 0.9984          | 0.9988     | 1.0000          |

### 5. Comparison of approximations for $\alpha = 53.43^\circ$

R. E. Miles (1969) provides yet another way of approximating  $P(N)$ . Theorem 3 of his paper yields, as a special case, the approximation

$$(10) \quad 1 - 2^{-N} N(N-1) \sin^2 \alpha (1 + \cos \alpha)^{N-2}$$

for  $P(N)$ . The table below compares values for  $\alpha = 53.43^\circ$  obtained from (10) with our simulation results. The best estimates of Moran and Fazekas de St. Groth, using (5) and (6), and the best estimates of Gilbert, using his lower bound (7), are included for comparison.

| $N$ | simulation<br>result | Miles   | Moran and<br>Fazekas de St. Groth | Gilbert |
|-----|----------------------|---------|-----------------------------------|---------|
| 10  | 0.0003               | -1.3842 | 0.0000                            | -2.1790 |
| 15  | 0.0465               | -0.7992 | 0.0154                            | -1.3989 |
| 20  | 0.2752               | -0.0529 | 0.1991                            | -0.4039 |
| 25  | 0.5763               | 0.4623  | 0.5084                            | 0.2831  |
| 30  | 0.7846               | 0.7479  | 0.7498                            | 0.6638  |
| 35  | 0.8993               | 0.8885  | 0.8857                            | 0.8513  |
| 40  | 0.9556               | 0.9527  | 0.9511                            | 0.9370  |
| 45  | 0.9837               | 0.9806  | 0.9799                            | 0.9741  |
| 50  | 0.9929               | 0.9922  | 0.9920                            | 0.9896  |
| 55  | 0.9974               | 0.9970  | 0.9969                            | 0.9959  |
| 60  | 0.9988               | 0.9988  | 0.9988                            | 0.9984  |

We can see that although Moran and Fazekas de St. Groth's candidate does the best for the smaller values of  $N$ , Miles' formula seems to be the most accurate for the higher values of  $N$ .

We note that each of the formulas appear to underestimate  $P(N)$ .

## 6. Comparison of approximations for $\alpha = \pi/2$

In order to provide another basis of comparison for the various approximation schemes, we have also calculated each method's estimate of  $P(N)$  for  $\alpha = \pi/2$  and  $N = 4, 5, 6, \dots, 25$ . We can judge the accuracy of the estimates by comparing them with the exact value of  $P(N)$  obtained from (1).

Let us first examine the four candidates from the paper of Moran and Fazekas de St. Groth. Recall that they derived (2) and (5) from different hypotheses, and that for each formula we can either use the exact value of  $E(Y^2)$  obtained by numerical integration or we can make use of the approximation (6). For  $N = 5, 10, 15, 20, 25$  (and  $\alpha = \pi/2$ ) the estimates of  $P(N)$  from these four methods, as well as the known values of  $P(N)$ , are displayed in the table below.

| $N$ | $P(N)$ | Value from<br>(2),(3)&(4) | Value from<br>(2) and (6) | Value from<br>(5),(3)&(4) | Value from<br>(5) and (6) |
|-----|--------|---------------------------|---------------------------|---------------------------|---------------------------|
| 5   | 0.3125 | -0.5009                   | -0.0897                   | 0.1157                    | 0.2470                    |
| 10  | 0.9102 | 0.8640                    | 0.8927                    | 0.8695                    | 0.8961                    |
| 15  | 0.9935 | 0.9915                    | 0.9928                    | 0.9915                    | 0.9928                    |
| 20  | 0.9996 | 0.9996                    | 0.9996                    | 0.9996                    | 0.9996                    |
| 25  | 1.0000 | 1.0000                    | 1.0000                    | 1.0000                    | 1.0000                    |

One can see that for  $\alpha = \pi/2$ , the best estimate for  $P(N)$  comes from using (5) and (6). This was also the case for  $\alpha = 53.43^\circ$ . Also note that each approximation formula underestimates  $P(N)$ , as was the case previously. Because there is only one uncovered region when  $\alpha = \pi/2$ , it is somewhat surprising that (5) approximates  $P(N)$  better than does (2), since (2) was derived under the hypothesis that there was only one vacant region.

For  $\alpha = \pi/2$  we can again use (7) to obtain a lower bound for  $P(N)$ , and we can use (8) to obtain an upper bound for  $P(N)$ . These bounds, both by Gilbert, and the exact value of  $P(N)$  are given below for various values of  $N$ .

| $N$ | lower bound (7) | known value (1) | upper bound (8) |
|-----|-----------------|-----------------|-----------------|
| 5   | 0.1667          | 0.3125          | 0.9688          |
| 10  | 0.8828          | 0.9102          | 0.9990          |
| 15  | 0.9915          | 0.9935          | 1.0000          |
| 20  | 0.9995          | 0.9996          | 1.0000          |
| 25  | 1.0000          | 1.0000          | 1.0000          |

In the table below we display the approximated values of  $P(N)$  derived from the work of Miles, equation (10). Also shown are the best approximations of Moran and Fazekas de St. Groth, (5) used with (6), and Gilbert's lower bound (7). The known values of  $P(N)$  from equation (1) are also given.

| $N$ | known<br>value | Miles  | Moran and<br>Fazekas de St. Groth | Gilbert |
|-----|----------------|--------|-----------------------------------|---------|
| 4   | 0.1250         | 0.2500 | 0.0903                            | 0.0000  |
| 6   | 0.5000         | 0.5313 | 0.4324                            | 0.3750  |
| 8   | 0.7734         | 0.7813 | 0.7360                            | 0.7083  |
| 10  | 0.9102         | 0.9121 | 0.8961                            | 0.8828  |
| 12  | 0.9673         | 0.9678 | 0.9629                            | 0.9570  |
| 14  | 0.9888         | 0.9889 | 0.9875                            | 0.9852  |
| 16  | 0.9963         | 0.9963 | 0.9959                            | 0.9951  |
| 18  | 0.9988         | 0.9988 | 0.9987                            | 0.9984  |
| 20  | 0.9996         | 0.9996 | 0.9996                            | 0.9995  |
| 22  | 0.9999         | 0.9999 | 0.9999                            | 0.9999  |
| 24  | 1.0000         | 1.0000 | 1.0000                            | 1.0000  |

It can be seen that the formula of Miles provides the most accurate estimates of  $P(N)$ , and also that Moran and Fazekas de St. Groth's candidate is better than Gilbert's. Note that Miles' formula overestimates  $P(N)$  for this value of  $\alpha$ , whereas before, with  $\alpha = 53.43^\circ$ , it underestimated  $P(N)$ . It is interesting to note that for  $\alpha = \pi/2$  Miles' formula reduces to

$$P(N) \approx 1 - 2^{-N}(N^2 - N)$$

which is very similar to the exact result,

$$P(N) = 1 - 2^{-N}(N^2 - N + 2).$$

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| 20. ABSTRACT (Continue on reverse side if necessary and identify by block number)<br>This paper describes the computer simulation of a coverage problem in geometric probability, that of placing random caps on the surface of a sphere. The simulation results were compared with exact values where known and the differences were negligible. This suggested the use of simulation results to assess several approximation formulas in the literature. |   |  |

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